General Logarithmic Corrections to Bekenstein-Hawking Entropy

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Abtract

Recently, there has been a lot of attention devoted to resolving the quantum corrections to the Bekenstein-Hawking entropy of the black hole. In particular, the coefficient of the logarithmic term in the black hole entropy correction has been of great interest. In this paper, the black hole is corresponded to a canonical ensemble in statistics by radiant spectrum, resulted from the black hole tunneling effect studies and the partition function of ensemble is derived. Then the entropy of the black hole is calculated. When the first order approximation is taken into account, the logarithmic term of entropy correction is consistent with the result of the generalized uncertainty principle. In our calculation, there are no uncertainty factors. The prefactor of the logarithmic correction and the one if fluctuation is considered are the same. Our result shows that if the thermal capacity is negative, there is no divergent term. We provide a general method for further discussion on quantum correction to Bekenstein-Hawking entropy. We also offer a theoretical basis for comparing string theory and loop quantum gravitaty and deciding which one is more reliable.

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1. Introduction

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One of the most remarkable achievements in gravitational physics was the realization that black holes have temperature and entropy [1 - 3]. There is a growing interest in the black hole entropy. Because entropy has statistical physics meaning in the thermodynamic system, it is related to the number of microstates of the system. However, in the general relativity of Einstein's theory, the black hole entropy is a pure geometrical quantity. If we compare the black hole with the general thermodynamics system, we can find out an important difference between them: the black hole is a void with strong gravitation, while an ordinary object is made up of atoms and molecules. Based on the microstructure of general thermodynamics system, we can use the statistic mechanics of microscopic components to explain the thermodynamic property of an object. However, whether the black hole has the inner freedom degree corresponding to its entropy is the key issue in the black hole's physics [4]. Let us suppose that the Bekenstein-Hawking entropy can be attributed a definite statistical meaning. Then how are these microstates defined or, in other words, how can they be counted? [5] This is a key problem in the black hole entropy. In recent years, string theory and loop quantum gravitaty both have been successful in statistically explaining the black hole entropy-area law [5]. Which one is more reliable? It is expected to make choice by discussing the quantum correction term of the black hole entropy. Therefore, studying the black hole entropy correction value becomes the focus of attention. Many ways of discussing the black hole entropy correction value have emerged [5-12]. (But over the last few years both are String Theory and in loop quantum gravity.)

Based on string theory and loop quantum gravitaty, the relationship of the black hole entropy-area is given by[13]

$$S = \frac{A}{4L_p^2} + \rho \ln \frac{A}{4L_p^2} + O\left(\frac{L_p^2}{A}\right), \tag{1}$$

where $A = 16\pi L_p^2 M^2$ is area of the black hole horizon, $L_p = \sqrt{\hbar G}$ is Planck length. For the case of Loop Quantum Gravity, which is here of interest, there is still no consensus on the coefficient of the logarithmic correction, ρ, but it is established [14-16] that there are no correction terms with stronger-than-logarithmic dependence on the area.

Most of the recent focus has been on ρ; that is, the coefficient of the leading-order correction or the logarithmic "prefactor". It has even been suggested that this particular parameter might be useful as a discriminator of prospective fundamental theories [16]. It is, therefore, appropriate to reflect upon the loop quantum gravity prediction of $\rho = -1/2$ (according to the most up-to-date rigorous calculation [17]); whereas string theory makes no similar type of assertion [5]. However, without any further input, how can we determine if any particular value of the prefactor is right or wrong? That is, unlike the tree-level calculations, this type of discrimination is based on a question to which we do not know the answer yet!

It becomes clear that, to proceed in this direction, one requires a method of fixing ρ indepent of the specific elements of any one particular model of quantum gravity. Recently Hawking radiation process of the black hole is given a new explanation-tunneling process. And the radiation spectrum of the black hole is obtained. Thereafter, using this radiant spectrum, we calculate the partition function of canonical ensemble and derive the correction term of the black hole entropy. In our calculation, there is no need to make any hypothesis. We provide a new method for the correction to the black hole entropy studies.

2. Calculation and analysis

Recently, Parikh, Wilczek and Kraus [18] discussed Hawking radiation by tunneling effect. They thought that tunnels in the process of the particle radiation of the black hole has no potential barrier before particles radiate. Potential barrier is produced by radiation particles itself. That is, during the process of tunneling effect creation, the energy of the black hole decrease and the radius of the black hole horizon reduces. The horizon radius becomes a new value that is smaller than the original value. The decrease of radius is determined by the value of energy of radiation particles. There is a classical forbidden band-potential barrier between original radius and the one after the black hole radiates. Parikh and Wilczek skillfully obtained the radiation spectrum of Schwarzschild and Reissner-Nordstrom black holes. Refs.[19-30] developed the method proposed by Parikh and Wilczek. They derived the radiation spectrum of the black hole in all kinds of space-time. Refs.[31-34] obtained radiation spectrum of Hawking radiation after considering the generalized uncertainty relation. However, Angheben, Nadalini, Vanzo and Zerbini calculated the radiation spectrum of arbitrary dimensional black hole and obtained that the general expression of the energy spectrum of Hawking radiation is as follows:

$$\rho_s \propto e^{\Delta S}, \tag{2}$$

where

$$\Delta S = S_{MC}(E-E_s) - S_{Mc}(E) = \sum_{k=1} \frac{1}{k!}\left(\frac{\partial^k S_{MC}(E_b)}{\partial E_b^k}\right)_{E_s=0} (-E_s)^k$$

$$= -\beta E_s + \beta_2 E_s^2 + \cdots, \tag{3}$$

$E_b = E - E_s$, according to thermodynamics relation, β should be the inverse of the temperature,

$$\beta_k = \frac{1}{k!}\left(\frac{\partial^k \ln \Omega}{\partial E_b^k}\right)_{s=0} = \frac{1}{k!}\left(\frac{\partial^k S_{MC}}{\partial E_b^k}\right)_{E_s=0}. \tag{4}$$

Normalizing ρ_s in (2), we obtain

$$\rho_s = \frac{1}{Z_c} e^{S_{MC}(E-E_s)-S_{MC}(E)},$$

where Z_c is called canonical partition function: $Z_c = \sum\limits_s e^{S_{MC}(E-E_s)-S_{MC}(E)}$ We begin with the formula for the canonical partition function of a classical system in equilibrium

$$Z_c(\beta) = \int_0^\infty e^{\Delta S} dE_s \rho(E_b), \tag{5}$$

where, $\rho(E_b)$ is the density of states. In what follows, we shall employ the identification $\rho(E_b) \equiv e^{S_{MC}(E_b)}$ [35], where, $S_{MC}(E_b)$ is the microcanonical entropy of an isolated subsystem whose energy is fixed at E_b. According to (3), the energy of the black hole radiant particles is E_s, then the energy of the black hole is $E_b = E - E_s$, for the black hole, the state density with energy E_b is $\rho(E - E_s)$. So

$$\rho(E-E_s) = \exp\left[S_{MC}(E-E_s)\right]. \tag{6}$$

The integral in Eq (5) can be performed in general by the saddle point approximation, provided the microcanonical entropy $S_{MC}(E - E_s)$ can be Taylor-expanded around the average equilibrium energy E,

$$S_{MC}(E-E_s) = S_{MC}(E) - \beta E_s + \beta_2 E_s^2 + \cdots, \tag{7}$$

and higher order terms in powers of the energy fluctuation represented by the ... in this expansion can neglected in comparison to the terms of second order.

(5) can be rewritten as

$$Z_c(\beta) = \int_0^\infty e^{-\beta E_s + \beta_2 E_s^2} dE_s e^{S_{MC}(E-E_s)} = e^{S_{MC}(E)} \int_0^\infty dE_s e^{-2\beta E_s + 2\beta_2 E_S^2}$$

$$= e^{S_{MC}(E)} \left[\frac{1}{2}\sqrt{\frac{\pi}{-2\beta_2}} \exp\left(\frac{\beta^2}{-2\beta_2}\right) \left(1 - erf\left(\frac{\beta}{\sqrt{-2\beta_2}}\right)\right)\right]. \tag{8}$$

where

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$$

is error integral.

Using the standard formula from equilibrium statistical mechanics

$$S = \ln Z_c - \beta \frac{\partial \ln Z_c}{\partial \beta}, \tag{9}$$

it is easy to deduce that the canonical entropy is given in terms of the microcanonical entropy by

$$S_C(E) = S_{MC}(E) + \Delta_S, \tag{10}$$

where

$$\Delta_S = \ln f(\beta) - \beta \frac{\partial \ln f(\beta)}{\partial \beta}, \tag{11}$$

$$f(\beta) = \frac{1}{2}\sqrt{\frac{\pi}{-2\beta_2}} \exp\left(\frac{\beta^2}{-2\beta_2}\right) \left[1 - erf\left(\frac{\beta}{\sqrt{-2\beta_2}}\right)\right]. \tag{12}$$

According to the asymptotic expression of error function

$$erf(z) = 1 - \frac{e^{-z^2}}{\sqrt{\pi} z} \left[1 + \sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!!}{(2z^2)^k}\right], \quad |z| \to \infty,$$

We have

$$f(\beta) = \frac{1}{2\beta}\left[1+\sum_{k=1}^{\infty}(-1)^k\frac{(2k-1)!!}{2^k}\left(\frac{\sqrt{-2\beta_2}}{\beta}\right)^{2k}\right]. \tag{13}$$

Substituting (13) into (11), we derive

$$\Delta_S = \ln\left[\frac{1}{2\beta}+\sum_{k=1}^{\infty}(-1)^k\frac{(2k-1)!!}{2^k 2\beta}\left(\frac{\sqrt{-2\beta_2}}{\beta}\right)^{2k}\right]$$

$$+\frac{1+\sum_{k=1}^{\infty}(-1)^k(2\sqrt{-2\beta_2})^{2k}\frac{(2k+1)(2k-1)!!}{2^k(2\beta)^{2k}}}{1+\sum_{k=1}^{\infty}(-1)^k(2\sqrt{-2\beta_2})^{2k}\frac{(2k-1)!!}{2^k(2\beta)^{2k}}}. \tag{14}$$

Using the definition of the thermal capacity of the system

$$C \equiv -\beta^2\left(\frac{\partial E}{\partial\beta}\right), \tag{15}$$

and

$$\beta_2 = -\frac{1}{2}\frac{\beta^2}{C}. \tag{16}$$

(14) can be expressed as

$$\Delta_S = \ln\left[T+T\sum_{k=1}^{\infty}(-1)^k\frac{(2k-1)!!}{2^k C^k}\right]+\frac{1+\sum_{k=1}^{\infty}(-1)^k\frac{(2k+1)(2k-1)!!}{2^k C^k}}{1+\sum_{k=1}^{\infty}(-1)^k\frac{(2k-1)!!}{2^k C^k}}. \tag{17}$$

where T is the temperature of the system. When we take the first order approximation, the logarithmic correction term of entropy is

$$\Delta_S = \ln T. \tag{18}$$

The correction to entropy is not related to thermal capacity.

For error function, if we take the sum of the series from the first term to n the term as the approximation of $erf(z)$. If z is real number, radical error does not exceed the absolute value of the first term in the series. So if

$C < -1$ or $C > 1$, it is sure that Δ_S is not divergent.

3. Conclusion

In summary, for Schwarzschild space-time, when we only take the first order approximation, the logarithmic correction term of entropy is

$$\Delta_S = \ln T = -\frac{1}{2}\ln\frac{A}{4} + const. \tag{19}$$

Using generalized uncertainty principle, Ref.[5] obtained that correction term of the black hole entropy was

$$S = \frac{A}{4} - \frac{\pi\alpha^2}{4}\ln\left(\frac{A}{4}\right) + \sum_{n=1}^{\infty} C_n \left(\frac{A}{4}\right)^{-n} + const. \tag{20}$$

From (20), we know that the logarithmic term of the black hole entropy correction contains uncertainty factor α^2. However, the uncertainty does not appear in our result.

After considering the correction to the black hole thermodynamics quantities due to thermal fluctuation, the expression of the entropy is [36-39]

$$S = \ln\rho = S_{MC} - \frac{1}{2}\ln(CT^2) + \cdots, \tag{21}$$

The above mentioned result is evidently limited. The thermal capacity of Schwarzschild black hole is negative causing the logarithmic correction term of the entropy in (21) to be divergent. So this relation is not valid for Schwarzschild black hole. However, any general four-dimensional curved space-time can be reduced to Schwarzschild space-time with proper approximation or limit. Hence (21) does not represent general property. However, in our result, the condition $C < -1$ or $C > 1$ can be satisfied by almost all black holes. So it is general.

In addition, previous results of the black hole entropy by other researchers were based on the fact that the black hole has thermal radiation and the radiation spectrum is a pure thermal one. However, Hawking radiation derives pure thermal spectrum under the condition that the space-time is invariant. The dispute over the loss of information arises while the radiation is dealt with. The black hole information loss means that pure quantum state will disintegrate to mixed state. This violates the unitary positive principle in quantum mechanics. Applying the tunneling effect method, we study the black hole radiation. Considering energy conservation and the change of horizon, the radiation spectrum is no longer a strictly pure thermal spectrum. This method avoids the limitation of Hawking radiation. It is pointed out that the self-gravity action provides the potential barrier of the quantum tunnel.

Our discussion is based on the quantum tunneling effect of the black hole radiation. Our result is very reasonable. We provide a new method for further studing the quantum correction to Bekenstein-Hawking entropy. We also offer a theoretical basis for comparing the string theory and loop quantum gravitaty and deciding which one is more reliable? Our result has shown that if the thermal capacity satisfies $-1 \leq C \leq 1$, the logarithmic correction term of the black hole entropy may be divergent. What does this divergent mean? This problem needs further investigation.

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